Truncated fractional moments of stable laws

John P. Nolan

American University

Abstract

Expressions are given for the truncated fractional moments $\text{EXP}_+^p$ of a general stable law. These involve families of special functions that arose out of the study of multivariate stable densities and probabilities. As a particular case, an expression is given for $E(X - a)_+$ when $\alpha > 1$.

Keywords: stable distribution, truncated moments, fractional moments

2000 MSC: 60E07, 60E10

1. Introduction

A univariate stable r.v. $Z$ with index $\alpha$, skewness $\beta$, scale $\gamma$, and location $\delta$ has characteristic function

$$
\phi(u) = \phi(u|\alpha, \beta, \gamma, \delta) = E \exp(iuZ) = \exp(-\gamma |u| + i\beta \eta(u, \alpha) + iu\delta), \quad (1)
$$

where $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $\gamma > 0$, $\delta \in \mathbb{R}$ and

$$
\eta(u, \alpha) = \begin{cases} 
- (\text{sign } u) \tan(\pi \alpha/2) |u| & \alpha \neq 1 \\
(2/\pi) u \ln |u| & \alpha = 1.
\end{cases}
$$

In the notation of Samorodnitsky and Taqqu (1994), this is a $S(\gamma, \beta, \delta)$ distribution. We will use the notation $X \sim S(\alpha, \beta, \gamma, \delta; 1)$ (the "$1" is used to distinguish between this parameterization and a continuous one used below).

The purpose of this paper is to derive expressions for truncated fractional moments $\text{EXP}_+^p = E(X I_{(X \geq 0)})^p$ for general stable laws. To do this, define the functions for real $x$ and $d$

$$
g_d(x|\alpha, \beta) = \begin{cases} 
\int_0^\infty \cos(xr + \beta \eta(r, \alpha)) r^{d-1} e^{-r} \, dr & 0 < d < \infty \\
\int_0^\infty [\cos(xr + \beta \eta(r, \alpha)) - 1] r^{d-1} e^{-r} \, dr & -2 \min(1, \alpha) < d \leq 0
\end{cases}
$$

Email address: jpnolan@american.edu (John P. Nolan)

1 The author was supported by an agreement with Cornell University, Operations Research & Information Engineering under W911NF-12-1-0385 from the Army Research Development and Engineering Command.

Preprint submitted to arXiv.org

March 21, 2018
The expressions for $E X^p_2$ will involve the functions $g_p(\cdot|\alpha, \beta)$ and $\tilde{g}_p(\cdot|\alpha, \beta)$, i.e. negative values fractional values of the subscript $d$. Before proving that result, we show that the functions $g_d(\cdot|\alpha, \beta)$ and $\tilde{g}_d(\cdot|\alpha, \beta)$ have multiple uses. For a standardized univariate stable law, Fourier inversion of the characteristic function shows that the d.f. and density are given by

$$F(x|\alpha, \beta) = \frac{1}{\pi} \left( g_0(x|\alpha, \beta) - \tilde{g}_0(0|\alpha, \beta) \right)$$

$$f(x|\alpha, \beta) = \frac{1}{\pi} g_1(x|\alpha, \beta).$$

We note that there are explicit formulas for $F(0|\alpha, \beta)$ when $\alpha \neq 1$.

The $g_d(\cdot|\alpha, \beta)$ functions are used in a similar way to give $d$-dimensional stable densities, see Theorem 1 of Abdul-Hamid and Nolan (1998) (note that there is a sign mistake in that formula when $\alpha = 1$), and Nolan (2017) uses both $g_d(\cdot|\alpha, \beta)$ and $\tilde{g}_d(\cdot|\alpha, \beta)$ to give an expression for multivariate stable probabilities.

Another use of these functions is in conditional expectation of $X_2$ given $X_1 = x$ when $(X_1, X_2)$ are jointly stable with zero shift and spectral measure $\Lambda$. In general, the conditional expectation is a complicated non-linear function; here it is restated in terms of these functions. If $\alpha > 1$ or ($\alpha \leq 1$ and (5.2.4) in Samorodnitsky and Taqqu (1994) holds), then Theorems 5.2.2 and 5.2.3 in Samorodnitsky and Taqqu (1994) show that the conditional expectation exists for $x$ in the support of $X_1$ and is given by

$$E(X_2|X_1 = x) =$$

$$\begin{cases} 
  c_1 x + c_2 \left[ \frac{1 - (x/\gamma_1)\tilde{g}_1(x/\gamma_1|\alpha, \beta_1)}{g_1(x/\gamma_1|\alpha, \beta_1)/\gamma_1} \right] & \alpha \neq 1 \\
  c_0 + c_1 \frac{x - \mu_1}{\gamma_1} + c_2 \frac{\tilde{g}_1((x - \mu_1)/\gamma_1 - (2\beta_1/\pi)\ln \gamma_1|1, \beta_1)}{g_1(x/\gamma_1|1, \beta_1)} & \alpha = 1, \beta_1 \neq 0 \\
  c_0 + c_1 \frac{x - \mu_1}{\gamma_1} + c_2 \left[ \frac{(1 - \ln \gamma_1)g_1((x - \mu_1)/\gamma_1|1, 0) + h_1((x - \mu_1)/\gamma_1|1, 0)}{g_1(x/\gamma_1|1, 0)} \right] & \alpha = 1, \beta_1 = 0,
\end{cases}$$

where $\beta_1$ and $\gamma_1$ are the skewness and scale parameters of $X_1$, and the constants and function $h_1(\cdot|1, 0)$ are given by

$$c_0 = -\frac{2}{\pi} \int_S s_2 \ln |s_1| \Lambda(ds)$$
\[ c_1 = \begin{cases} \frac{\kappa_1 + \beta_1 \tan(\pi \alpha/2) \kappa_2}{\kappa_2 (1 + \beta_1^2 \tan^2(\pi \alpha/2))} & \alpha \neq 1 \\ \frac{\kappa_1}{\kappa_1} & \alpha = 1, \beta_1 \neq 0 \\ \kappa_2/\beta_1 & \alpha = 1, \beta_1 = 0 \end{cases} \]

\[ c_2 = \begin{cases} \frac{\tan(\pi \alpha/2)(\kappa_2 - \beta_1 \kappa_1)}{\kappa_1 (1 + \beta_1^2 \tan^2(\pi \alpha/2))} & \alpha \neq 1 \\ \frac{\kappa_2 - \beta_1 \kappa_1}{\beta_1} & \alpha = 1, \beta_1 \neq 0 \\ -2\kappa_2/\pi & \alpha = 1, \beta_1 = 0 \end{cases} \]

\[ \kappa_1 = [X_2, X_1] = \begin{cases} \int_{\mathbb{S}} s_2 s_1^{-1} \Lambda(ds) & \alpha \neq 1 \\ \int_{\mathbb{S}} s_2 s_1^{0} \Lambda(ds) = \int_{\mathbb{S}} s_2 \text{sign}(s_1) \Lambda(ds) & \alpha = 1 \end{cases} \]

\[ \kappa_2 = \frac{1}{\pi} \int_{\mathbb{S}} s_1 \ln |s_1| \Lambda(ds) \]

\[ \mu_1 = -\frac{2}{\pi} \int_{\mathbb{S}} s_1 \ln |s_1| \Lambda(ds) \]

\[ h(x|1,0) = \int_0^\infty \cos(xr)(\log r)e^{-r}dr. \]

In the terms above, \( \mathbb{S} \) is the unit circle and \([X_2, X_1] \) is the \( \alpha \)-covariation. Note that if \( \Lambda \) is symmetric, then \( c_0 = \kappa_2 = \beta_1 = \mu_1 = 0 \), so \( c_2 = 0 \) and

\[ E(X_2|X_1 = x) = \frac{[X_2, X_1]}{\kappa_1} x \]

is linear.

### 2. Truncated moments \( EX_+^p \)

The main result of this paper is the following expression for the fractional truncated moment of a stable r.v.

**Theorem 1.** Let \( X \sim \mathbb{S}(\alpha, \beta, \gamma, \delta; 1) \) with any \( 0 < \alpha < 2 \) and any \(-1 \leq \beta \leq 1 \) and set

\[ \delta^* = \begin{cases} \delta/\gamma & \alpha \neq 1 \\ \delta/\gamma + (2/\pi)\beta \log \gamma & \alpha = 1 \end{cases} \]

For \( p < \alpha \), define \( m^p(\alpha, \beta, \gamma, \delta) = EX_+^p \).

(a) When \( p = 0 \),

\[ m^0(\alpha, \beta, \gamma, \delta) = P(X > 0) = \frac{1}{2} - \frac{1}{\pi} \tilde{g}_0(-\delta^*|\alpha, \beta). \]

When \( 0 < p < \min(1, \alpha) \),

\[ m^p(\alpha, \beta, \gamma, \delta) = \gamma^p \frac{\Gamma(p+1)}{\pi} \left[ \sin(\frac{\pi p}{\alpha}) \left( \frac{\Gamma(1-p/\alpha)}{p} - \tilde{g}_p(-\delta^*|\alpha, \beta) \right) - \cos(\frac{\pi p}{\alpha}) \tilde{g}_p(-\delta^*|\alpha, \beta) \right]. \]
When $p = 1 < \alpha < 2$,

$$m^p(\alpha, \beta, \gamma, \delta) = \gamma \left[ \frac{\delta^*}{2} + \frac{1}{\pi} \Gamma(1 - 1/\alpha) - g_{-1}(-\delta^*|\alpha, \beta) \right].$$

When $1 < p < \alpha < 2$,

$$m^p(\alpha, \beta, \gamma, \delta) = \gamma^p \frac{\Gamma(p+1)}{\pi} \left[ \sin \left( \frac{\delta^*}{p} \right) \left( \frac{\Gamma(1 - p/\alpha)}{p} - g_{-p}(-\delta^*|\alpha, \beta) \right) 
\quad + \cos \left( \frac{\delta^*}{\alpha} \right) \left( \frac{\delta^*}{\alpha} \Gamma((1 - p)/\alpha) - \bar{g}_{-p}(-\delta^*|\alpha, \beta) \right) \right].$$

(b) $EX^p = E(-X)^p = m^p(\alpha, -\beta, \gamma, -\delta)$.

**Proof**  

(a) To simplify calculations, first assume $\gamma = 1$; the adjustment for $\gamma \neq 1$ is discussed below. When $p = 0$, $EX^0 = \int_0^\infty 1 f(x)dx = P(X > 0)$, and (2) and $\bar{g}_0(x|\alpha, \beta) \to \pi/2$ as $x \to \infty$ gives the value in terms of $\bar{g}_0(-\alpha, \beta)$.

When $0 < p < \min(1, \alpha)$, Corollary 2 of Pinelis (2011) with $k = \ell = 0$ shows

$$EX^p = \frac{\Gamma(p+1)}{\pi} \int_0^\infty \Re \frac{\phi(u) - 1}{(iu)^{p+1}} du. \tag{3}$$

First assume $\alpha \neq 1$ and set $\zeta = \zeta(\alpha, \beta) = -\beta \tan \theta/2$ and restricting to $u > 0$,

$$\frac{\phi(u) - 1}{(iu)^{p+1}} = \left[ \left( e^{-u} (1+i\zeta)+i\delta u - 1 \right) (-i) e^{-i(\pi/2)p} \right] u^{-p-1}$$

$$= \left[ -i \left( e^{-u} \left( \cos(\delta u - \zeta u) + i \sin(\delta u - \zeta u) \right) - 1 \right) e^{-i(\pi/2)p} \right] u^{-p-1}$$

$$= \left[ e^{-u} \sin(\delta u - \zeta u) - i \left( e^{-u} \cos(\delta u - \zeta u) - 1 \right) \right] \left( \cos \left( \frac{\delta^*}{2} \right) - i \sin \left( \frac{\delta^*}{2} \right) \right) u^{-p-1}$$

And therefore

$$\Re \frac{\phi(u) - 1}{(iu)^{p+1}} = \left[ \cos \left( \frac{\delta^*}{2} \right) e^{-u} \sin(\delta u - \zeta u) - \sin \left( \frac{\delta^*}{2} \right) e^{-u} \cos(\delta u - \zeta u) - 1 \right] u^{-p-1}$$

$$= \cos \left( \frac{\delta^*}{2} \right) \sin(\delta u - \zeta u) u^{-p-1} e^{-u}$$

$$- \sin \left( \frac{\delta^*}{2} \right) \left[ \cos(\delta u - \zeta u) - 1 \right] u^{-p-1} e^{-u} + (e^{-u} - 1)u^{-p-1}$$

Integrating this from 0 to $\infty$, substituting $t = u$ in the last term to get

$$EX^p = \frac{\Gamma(p+1)}{\pi} \left[ - \cos \left( \frac{\delta^*}{2} \right) \bar{g}_{-p}(-\delta|\alpha, \beta) - \sin \left( \frac{\delta^*}{2} \right) \left( g_{-p}(-\delta|\alpha, \beta) - \Gamma(1 - p/\alpha)/p \right) \right].$$

Next consider $0 < p < \alpha = 1$. Use (3) again, so we need to simplify

$$\frac{\phi(u) - 1}{(iu)^{p+1}} = \left[ e^{-u(1+i\eta(u, 1)) + i\delta u} - 1 \right] (-i) e^{-i(\pi/2)p} u^{-p-1}$$
\[
\begin{align*}
\phi(u) - 1 - iu\delta & = \left(-i \left(e^{-u} \left[\cos(\delta u - \beta \eta(u, 1)) + i \sin(\delta u - \beta \eta(u, 1))\right] - 1\right) e^{-i(\pi/2)p}\right) u^{-p-1} \\
& = \left(\left[e^{-u} \sin(\delta u - \beta \eta(u, 1)) - i \left(e^{-u} \cos(\delta u - \beta \eta(u, 1)) - 1\right)\right]\times \left[\cos(\phi) - i \sin(\phi)\right]\right) u^{-p-1}
\end{align*}
\]
\[
\mathbb{R} \frac{\phi(u) - 1}{(iu)^{p+1}} = \left[\cos(\phi) e^{-u} \sin(\delta u - \beta \eta(u, 1)) - \sin(\phi) \left(e^{-u} \cos(\delta u - \beta \eta(u, 1)) - 1\right)\right] u^{-p-1}.
\]

Integrating from 0 to \(\infty\) yields
\[
EX_p^+ = \frac{\Gamma(p+1)}{\pi} \left[ - \cos(\phi) g_{-p}(-\delta|1, \beta) - \sin(\phi) \left\{ g_{-p}(-\delta|1, \beta) - \Gamma(1-p)/p \right\} \right].
\]

When \(p = 1 < \alpha < 2\), \(EX\) exists and is equal to \(\delta\). Using Corollary 2 of [Pinelis (2011)] with \(k = 1\), \(\ell = 0\) shows
\[
EX_+ = \frac{1}{2}EX + \frac{\Gamma(2)}{\pi} \int_0^\infty \mathbb{R} \frac{\phi(u) - 1}{(iu)^{p+1}} du = \frac{\delta}{2} + \frac{1}{\pi} \int_0^\infty \mathbb{R} \frac{\phi(u) - 1}{(iu)^{p+1}} du.
\]
The integrand is the same as above, with \(\cos(\phi) = 0\) and \(\sin(\phi) = 1\), so
\[
EX_+ = \frac{\delta}{2} - \frac{1}{\pi} \left[ g_{-1}(-\delta|\alpha, \beta) - \Gamma(1-1/\alpha) \right].
\]

When \(1 < p < \alpha < 2\), Corollary 2 of [Pinelis (2011)] with \(k = \ell = 1\) shows
\[
EX_p^+ = \frac{\Gamma(p+1)}{\pi} \int_0^\infty \mathbb{R} \frac{\phi(u) - 1 - iuEX}{(iu)^{p+1}} du, \quad (4)
\]

Since \(\alpha > 1\), \(EX\) exists and is equal to \(\delta\). As above, for \(u > 0\),
\[
\phi(u) - 1 - iu\delta = \left(\left[e^{-u} \left[\cos(\delta u - \zeta u) + i \sin(\delta u - \zeta u)\right] - 1\right] + i \delta u\right) \left(-i\right) e^{-i(\pi/2)p}\right) u^{-p-1} \\
= \left(-i \left(e^{-u} \left[\cos(\delta u - \zeta u) + i \sin(\delta u - \zeta u)\right] - 1\right) - i \delta u\right) e^{-i(\pi/2)p}\right) u^{-p-1} \\
= \left[\left(e^{-u} \sin(\delta u - \zeta u) - \delta u\right) - \left(e^{-u} \cos(\delta u - \zeta u) - 1\right)\right] u^{-p-1}
\]
\[
\times \left[\cos(\phi) - i \sin(\phi)\right] u^{-p-1}
\]
And therefore
\[
\mathbb{R} \frac{\phi(u) - 1 - iu\delta}{(iu)^{p+1}} = \left[\cos(\phi) \left(e^{-u} \sin(\delta u - \zeta u) - \delta u\right) - \sin(\phi) \left(e^{-u} \cos(\delta u - \zeta u) - 1\right)\right] u^{-p-1} \\
= \cos(\phi) \left[\sin(\delta u - \zeta u) - \delta u\right] u^{-p-1} e^{-u} + \delta(e^{-u} - 1)u^{-p}\right) \\
- \sin(\phi) \left[\cos(\delta u - \zeta u) - 1\right] u^{-p} e^{-u} + (e^{-u} - 1)u^{-p-1}\right)
\]

5
Corollary (b) This follows from
\[ S \]
the closed form expressions for \( m \).

Corollary The previous result shows the following.

3. Plugging this into (4) and integrating yields
\[ EX^p = \frac{\Gamma(p+1)}{\pi} \left\{ \cos(\frac{\phi}{\pi})[-g_p(-\delta|\alpha, \beta)] + (\delta/\alpha)(\Gamma((1-p)/\alpha)] \right. \\
\left. + \sin(\frac{\phi}{\pi})[-g_p(-\delta|\alpha, \beta) + \Gamma(1-p/\alpha)/p] \right\} \]

Now consider \( \gamma \neq 1 \). If \( X \sim S(\alpha, \beta, \gamma, \delta; 1) \), then \( X \sim S(\alpha, \beta, 1, \delta^*; 1) \), so \( EX^p = \gamma^p EY^p \). In symbols,
\[ m^p(\alpha, \beta, \gamma, \delta) = \gamma^p m^p(\alpha, \beta, 1, \delta^*) \]
(b) This follows from \( -X \sim S(\alpha, -\beta, \gamma, -\delta; 1) \).

When \( -1 < p < 0 \), we conjecture that
\[ m^p(\alpha, \beta, \gamma, \delta) = \gamma^p \frac{\Gamma(p+1)}{\pi} [-\sin(\frac{\phi}{\pi})g_p(-\delta^*|\alpha, \beta) - \cos(\frac{\phi}{\pi})g_p(-\delta^*|\alpha, \beta)] \]

3. Related results

There are several corollaries to the preceding result. First, taking \( p = 1 \) in the previous result shows the following.

Corollary 2. If \( X \sim S(\alpha, \beta, \gamma, \delta; 1) \) with \( \alpha > 1 \), \( -1 \leq \beta \leq 1 \), \( a \in \mathbb{R} \)
\[ E(X-a)_+ = \frac{\delta-a}{2} + \frac{\gamma}{\pi} \left[ \Gamma \left( 1 - \frac{1}{\alpha} \right) - g_1 \left( \frac{\delta-a}{\gamma} \alpha, \beta \right) \right] \]

Combining parts (a) and (b) of Theorem\[ \square \]yields.

Corollary 3. If \( X \sim S(\alpha, \beta, \gamma, \delta; 1) \) with \( 0 < \alpha < 2 \), \( -1 \leq \beta \leq 1 \), \( -1 \leq p < \alpha \).
\[ E|X|^p = \gamma^p \frac{2\Gamma(p+1)}{\pi} \sin(\frac{\phi}{\pi})\left( \frac{\delta \Gamma(1-p/\alpha)}{p} \mathbb{1}_{p>0} - g_p(-\delta^*|\alpha, \beta) \right) \]
\[ EX^{<p>} = \gamma^p \frac{2\Gamma(p+1)}{\pi} \cos(\frac{\phi}{\pi})\left( \frac{\delta^* \Gamma((1-p)/\alpha)}{\alpha} \mathbb{1}_{p>1} - g_p(-\delta^*|\alpha, \beta) \right) \]

Proof \( E|X|^p = EX^p + EX^p = m^p(\alpha, \beta, \gamma, \delta) + m^p(\alpha, -\beta, \gamma, -\delta) \) and \( EX^{<p>} = m^p(\alpha, \beta, \gamma, \delta) - m^p(\alpha, -\beta, \gamma, -\delta) \). Use Theorem\[ \square \]and the reflection property:
\( g_d(-x|\alpha, \beta) = g_d(x|\alpha, -\beta) \). Note that as \( p \to 0 \), \( E|X|^p \to E1 = 1 \) and \( EX^{<p>} \to -2\pi g_0(\delta^*|\alpha, \beta) = P(X > 0) - P(X < 0) = 1 - 2F(0) \). Also as \( p \to 1 \), \( EX^{<p>} \to \delta \). \[ \square \]

In the strictly stable case, the expressions for \( EX^p \) can be simplified using closed form expressions for \( g_d(0|\alpha, \beta) \) and \( g_d(0|\alpha, \beta) \) when \( \alpha \neq 1 \). To state the result, set
\[ \theta_0 = \theta_0(\alpha, \beta) = \begin{cases} \alpha^{-1} \arctan(\beta \tan \frac{\phi}{\pi}) & \alpha \neq 1 \\ \pi/2 & \alpha = 1. \end{cases} \]
Lemma 4. When $\alpha \neq 1$,
\[
g_d(0|\alpha, \beta) = \begin{cases} 
(c \alpha \theta_0)^{d/\alpha} \cos(d \theta_0) \Gamma(1 + d/\alpha)/d & d > 0 \\
\ln(c \alpha \theta_0)/\alpha & d = 0 \\
[(c \alpha \theta_0)^{d/\alpha} \cos(d \theta_0) - 1] \Gamma(1 + d/\alpha)/d & -\alpha < d < 0 \\
-\frac{(c \alpha \theta_0)^{d/\alpha} \sin(d \theta_0) \Gamma(1 + d/\alpha)/d}{-\theta_0} & d \in (-\alpha, 0) \cup (0, \infty) 
\end{cases}
\]
\[
\tilde{g}_d(0|\alpha, \beta) = \begin{cases} 
(c \alpha \theta_0)^{d/\alpha} \cos(d \theta_0) \Gamma(1 + d/\alpha)/d & d > 0 \\
\ln(c \alpha \theta_0)/\alpha & d = 0 \\
-\frac{(c \alpha \theta_0)^{d/\alpha} \sin(d \theta_0) \Gamma(1 + d/\alpha)/d}{-\theta_0} & d \in (-\alpha, 0) \cup (0, \infty) 
\end{cases}
\]

Proof Substitute $u = r$ in the expressions for $g_d(0|\alpha, \beta)$ and $\tilde{g}_d(0|\alpha, \beta)$. Then use respectively the integrals 3.944, 3.948.2, 3.945.1, 3.944.5, and 3.948.1 pg. 492-493 of Gradshteyn and Ryzhik (2000). (Note that some of these formulas have mistyped exponents.) Finally, when $\alpha \neq 1$, $c \theta_0 = -\arctan \zeta$, and for the allowable values of $\alpha$ and $\theta_0$,
\[
\cos(c \theta_0) = |\cos(c \theta_0)| = (1 + \tan^2 c \theta_0)^{-1/2} = (1 + \zeta^2)^{-1/2}.
\]

The following is a different proof of Theorem 2.6.3 of Zolotarev (1986).

Corollary 5. Let $X$ be strictly stable, e.g. $X \sim S(\alpha, \beta, \gamma, 0; 1)$ with $\alpha \neq 1$ or ($\alpha = 1$ and $\beta = 0$) and $0 < p < \alpha$.
(a) The fractional moment of the positive part of $X$ is
\[
EX^p_+ = \frac{\gamma^p}{(\cos(c \theta_0)^{p/\alpha}} \frac{\Gamma(1 - p/\alpha) \sin(p/2 + \theta_0)}{\Gamma(1 - p)} \frac{\sin(p\pi)}{\sin(p\pi)}.
\]
(b) The fractional moment of the negative part of $X$ is $EX^p_- = EX^p_+(-X)^p_+$, which can be obtained from the right hand side above by replacing $\theta_0$ with $-\theta_0$.

When $p = 1$, the product $\Gamma(1 - p)\sin(p\pi)$ in the denominator above is interpreted as the limiting value as $p \to 1$, which is $\pi$.

Proof Note that when $X$ is strictly stable, $\delta^* = 0$. First assume $0 < p < \min(1, \alpha)$ and substitute Lemma 4 into this case of Theorem 1.
\[
EX^p_+ = \frac{\gamma^p \Gamma(p + 1)}{\pi} \left[ \sin(p\pi/2) \left( \frac{\Gamma(1 - p/\alpha)}{p} - \left( (\cos(c \theta_0)^{-p/\alpha} \cos(-p\theta_0) - 1 \right) \frac{\Gamma(1 - p/\alpha)}{-p} \right) \right.
\]
\[
\left. - \cos(p\pi/2)(- \cos(c \theta_0)^{-p/\alpha} \sin(-p\theta_0) \frac{\Gamma(1 - p/\alpha)}{-p} \right],
\]
\[
= \frac{\gamma^p \Gamma(p + 1) \Gamma(1 - p/\alpha)}{\pi p \cos(c \theta_0)^{p/\alpha}} \left[ \sin(p\pi/2) \cos(p\theta_0) + \cos(p\pi/2) \sin(p\theta_0) \right]
\]
\[
= \frac{\gamma^p \Gamma(p + 1) \Gamma(1 - p/\alpha)}{\pi p \cos(c \theta_0)^{p/\alpha}} \sin(p\pi/2 + p\theta_0)
\]
Using the identity \( \Gamma(p + 1) = \pi p/\Gamma(1 - p) \sin p\pi \) gives the result.

When \( p = 1 < \alpha \), again using the appropriate part of Theorem 1 shows

\[
EX_+ = \gamma \left[ 0 + \frac{1}{\pi} \left( \Gamma(1 - 1/\alpha) - \left( \cos \alpha \theta_0 \right)^{-1/\alpha} \cos(-\theta_0) - 1 \right) \frac{\Gamma(1 - 1/\alpha)}{-1} \right]
\]

\[
= \frac{\gamma \Gamma(1 - 1/\alpha)}{\pi} \left( \cos \alpha \theta_0 \right)^{-1/\alpha} \cos(-\theta_0).
\]

When \( 1 < p < \alpha \), using Theorem 1 and \( \delta^* = 0 \),

\[
EX_p = \gamma^p \Gamma(p + 1) \left[ \sin(\pi p/2) \left( \frac{\Gamma(1 - p/\alpha)}{p} - \left( \cos \alpha \theta_0 \right)^{-p/\alpha} \cos(-p\theta_0) - 1 \right) \frac{\Gamma(1 - p/\alpha)}{-p} \right]
\]

\[
+ \cos(\pi p/2) \left( 0 - \cos(\alpha \theta_0) \right)^{-p/\alpha} \sin(-p\theta_0) \frac{\Gamma(1 - p/\alpha)}{-p},
\]

and the rest is like the first case.

The standard parameterization used above is discontinuous in the parameters near \( \alpha = 1 \), and it is not a scale-location family when \( \alpha = 1 \). To avoid this, a continuous parameterization that is a scale-location family can be used. We will say \( X \sim S(\alpha, \beta, \gamma, \delta; 0) \) if it has characteristic function

\[
E \exp(iuX) = \begin{cases} 
\exp(-\gamma |u| [1 + i\beta(\tan \frac{\gamma}{2\pi})(\text{sign } \gamma)(|\gamma u|^{1/\alpha} - 1) + i\delta u)] & \alpha \neq 1 \\
\exp(-\gamma |u| [1 + i\beta(2/\pi)(\text{sign } \gamma)(|\gamma u|) + i\delta u]) & \alpha = 1.
\end{cases}
\]

A stable r. v. \( X \) can be expressed in both the 0-parameterization and the 1-parameterization, in which case the index \( \alpha \), the skewness \( \beta \) and the scale \( \gamma \) are the same. The only difference is in the location parameter: if \( X \) is simultaneously \( S(\alpha, \beta, \gamma, \delta_0; 0) \) and \( S(\alpha, \beta, \gamma, \delta_1; 1) \), then the shift parameters are related by

\[
\delta_1 = \begin{cases} 
\delta_0 - \beta \gamma \tan \frac{\gamma}{2\pi} & \alpha \neq 1 \\
\delta_0 - (2/\pi)\beta \gamma \log \gamma & \alpha = 1.
\end{cases}
\]

Therefore, if \( X \sim S(\alpha, \beta, \gamma, \delta_0; 0) \),

\[
EX_+^p = \begin{cases} 
m^p(\alpha, \beta, \gamma, \delta_0 - \beta \gamma \tan \frac{\gamma}{2\pi}) & \alpha \neq 1 \\
m^p(\alpha, \beta, \gamma, \delta_0 - (2/\pi)\beta \gamma \log \gamma) & \alpha = 1.
\end{cases}
\]

This quantity is continuous in all parameters.

For the above expressions for \( EX_+^p \) to be of practical use, one must be able to evaluate \( g_d(|\alpha, \beta) \) and \( \tilde{g}_d(|\alpha, \beta) \). When \( d \) is a nonnegative integer, Nolan (2017) gives Zolotarev type integral expressions for these functions. However, this is not helpful here, where negative, non-integer values of \( d \) are needed. We have written a short R program to numerically evaluate the defining integrals for \( g_d(|\alpha, \beta) \) and \( \tilde{g}_d(|\alpha, \beta) \). A single evaluation takes less than 0.0002 seconds on a modern desktop. This faster than numerically evaluating \( EX_+^p = \int_0^\infty x^p f(x | \alpha, \beta, \gamma, \delta) dx \), because the latter requires many numerical calculations of the density \( f(x | \alpha, \beta, \gamma, \delta) \).
References


