Optimal Execution with a VWAP Benchmark

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Optimal Liquidation is a fundamental problem for investment banks as well as their clients.

Important for both principal and agency trading.

The problem is typically formulated so as to minimize the sum of execution cost (slippage) and risk (understood as the variance of execution cost).

Most solutions based upon (generalizations of) the seminal work by Almgren & Chriss.

Benchmark chosen is the order arrival price (midquote).
Choice of Benchmark

- **Typical setup** - Client provides a quantity (shares) to buy or sell - \( Y \) together with a start and end time and the goal of the algorithm is to match the market VWAP over the period.

- Market VWAP is defined as follows, given a sequence of trades executed at prices \( P_1, \ldots, P_N \) with quantities \( V_1, \ldots, V_N \) it is given by

\[
\bar{P} = \frac{\sum P_i V_i}{\sum V_i}
\]

- We thus want to formulate a problem, similar to the classical Almgren and Chriss setting, i.e.

\[
\min_u \mathbb{E}[S(u)] + \lambda \mathbb{V}[S(u)]
\]

where now expected slippage and variance of slippage are relative to a *VWAP benchmark* and we make the definition

\[
S(u) = Y(P^u - \bar{P})
\]

where \( P^u \) is the average price per share realised on the order executed with strategy \( u \).
VWAP Benchmark

- Popular as it is considered a fair measure of what the average uninformed trader can achieve. It is believed to be difficult to achieve very good and very bad performance against VWAP.
- What is important is not the absolute volume, but the relative volume. If you allocate your trades in the same fraction that the market does, you will get good performance.
- This problem is more involved than the classical Almgren and Chriss set-up. Arrival price is a known deterministic value. Market VWAP is a stochastic benchmark.
- Clients generally do not like variance in their execution - it is often in their interest to accept slightly worse average performance (slippage) in return for less variance/standard deviation around that mean. Less uncertainty in their investment decisions.
Model Components

- We assume the quantity to be bought to be $Y$ and that our stock holdings evolve according to the SDE.

$$dX^u(t) = u(t)dt, X(T) = Y.$$  

- The process $u$ is typically called trading rate.

- We assume an unperturbed mid-quote price $P$ given by

$$P(t) = P(0) + \sigma W(t)$$

where $\sigma$ is the daily volatility in currency units.

- When we trade, due to costs we execute at a price given by

$$\tilde{P}(t) = P(t) + \kappa u(t).$$

Here the coefficient $\kappa$ reflects our impact on the market (and arises from a linear temporary market impact model).
For simplicity we look at day VWAP orders (very common) & fix start time as 0 and end time as $T$. We need a continuous time approximation to the day VWAP price.

$$\bar{P} = \frac{\sum P_i V_i}{\sum V_i} = \frac{\sum P_i \Delta V_i}{\sum V_i} = \int_0^T P(t) d\left(\frac{V(t)}{V(T)}\right).$$

We need to model the relative volume process $\frac{V(t)}{V(T)}$.

Note the use of $P$ not $\tilde{P}$, simplification. Two potential justifications:

- Average sizes are usually very small ($\leq 5\%$ ADV).
- There is often interest in looking at market VWAP net of our own actions.

Difficult to relax since then you would see effects in the denominator $V(T)$ of your control $u$. 
Relative Volume Process

- There are natural constraints the relative volume (denoted by $\gamma$) must satisfy, namely $\gamma(0) = 0$, $\gamma(T) = 1$ and it must be increasing.
- A natural stochastic process for modelling this is the Gamma Bridge, this is obtained by taking a gamma process and dividing by its terminal value.
- There is a known form for the generator and $\gamma(t)$ is a Dirichlet process.
- If the volume has independent increments (unreasonable) and is independent of the terminal value (more reasonable) then it can be shown that relative volume must be a gamma bridge.
- Note that this process requires knowledge of the day’s volume so is not adapted to the filtration generated by the price and cumulative volume (gamma process).
Final Optimization Problem

- Assuming a gamma bridge for relative volume, one can show (under mild assumptions) that the expected slippage can be written (for a strategy $u$) as

$$\mathbb{E}[S(u)] = \mathbb{E}\left[\int_0^T \kappa u^2(t) dt\right]$$

- Additionally that the variance of slippage satisfies

$$\mathbb{V}[S(u)] \approx \sigma^2 \mathbb{E}\left[\int_0^T (X^u(t) - Y\gamma(t))^2 dt\right]$$

- This approximation leads (in numerical simulations) to a relative error of 0.1%, it is necessary to make the problem tractable, dynamizing this leads to an optimal control problem of the form.

$$v(t, x, \gamma) = \inf_u \mathbb{E}\left[\int_t^T \kappa u^2(s) ds + \lambda \sigma^2 \int_t^T (X^u(s) - Y\gamma(s))^2 ds \bigg| \mathcal{F}(t)\right].$$

with $X(T) = Y$. 
Main Objection to Model Formulation

- One does not know the value $V(T)$ at the beginning of the day!
- Provides a benchmark against which we can test different strategies and compare them.
- Allows us to quantify the effect new estimators of $V(T)$ would have on VWAP performance.
- Helps assign a value to the knowledge of terminal volume in VWAP.
- Can we simply plug in an estimator of relative volume?

1Disclaimer: This is a theoretical formulation of how to trade a VWAP order and does not necessarily bear any resemblance to how DB actually implements the VWAP strategy.
HJB PDE

The problem would be a regular LQP problem if it were not for the completion constraint, \( X(T) = Y \). This introduces a singular terminal condition of the form

\[
\psi(T, x, \gamma) = \begin{cases} 
\infty & \text{if } x \neq Y \\
0 & \text{otherwise}.
\end{cases}
\]

The HJB PDE is given by

\[
\psi_t + \lambda \sigma^2 (\gamma - x)^2 + \inf_{u \in \mathbb{R}} (\psi_x u + \kappa u^2) \\
+ m \int_0^1 (\psi(t, x, \gamma + (1 - \gamma)z) - \psi(t, x, \gamma)) (1 - z) \frac{1}{z} dz = 0
\]

This can be solved in a few easy steps, first replace the singular condition by a term of the form \( n(X(T) - Y)^2 \). Then take limits in the solution and finally prove a verification theorem using the candidate limiting solution.
Main Results

Theorem (Frei & W. 2013 Theorem 3.1)

*The HJB equation corresponding to the control problem has a unique $C^{1,2}$ solution which can be written down in closed form. There is a unique optimal Markov control $\hat{u}(t, x, \gamma)$.***

Theorem (Frei & W. 2013 Theorem 3.1)

*There exists a frontier function $\zeta(t, \gamma)$, such that for each $t$, $\hat{u}(t) < 0$ when $\hat{X}(t) > \zeta(t, \gamma(t))$ and $\hat{u}(t) > 0$ when $\hat{X}(t) < \zeta(t, \gamma(t))$. That is to say the $x$-plane is divided into buy and sell regions.*

![An important corollary of this is - if $\hat{X}(0) = 0$ the optimal control satisfies $\hat{u}(t, \hat{X}(t), \gamma(t)) \geq 0$ for all $t \in [0, T]$. I.e it is always positive^2.](Note)

^2FINRA directive 5310 on Best Execution and Interpositioning
Simulations of $\hat{X}$ for different starting points $(t, \hat{X}(t))$ while the path of the gamma bridge is always the same. $\hat{X}$ crosses the frontier $\zeta$ (black curve) from above.
Simulations of $\hat{X}$ for different starting points $(t, \hat{X}(t))$ while the path of the gamma bridge is always the same. $\hat{X}$ does not cross the frontier $\zeta$ (black curve) from below.
Decomposition of Optimal Control

Theorem (Frei & W. 2013 Corollary 3.4)

The optimal control \( \hat{u}(t, x, \gamma) \) can be written as

\[
\hat{u}(t, x, \gamma) = \hat{u}_1(t) + \hat{u}_2(t, x, \gamma)
\]

where \( u_1 \) is a process satisfying \( \hat{u}_1(t) = \partial_t \mathbb{E}[\gamma(t)] \).

- Think of this as optimal = historical mean + adaptivity component, exactly how one would expect it to be.
- Justifies the statement that historical mean based approach should approximately optimal.
Gamma Bridge

- Gamma bridge is such that the mean satisfies
  \[ \mathbb{E}[\gamma(t)] = \frac{t}{T}. \]

- This is clearly unreasonable, relative volume is known to inherit the S-shape from the intraday volume profile.

- **Solution - Use a deterministic time change!**

- Choose a function \( G : [0, T] \mapsto [0, T] \) which is increasing and of the form
  \[ G(t) = at^3 + bt^2 + ct + d \]

  and observe we need \( G(0) = 0, G(T) = T \) so that there are only 2 free parameters (+1 from the gamma bridge).

- Now estimate using method of moments on 60 day mean relative intraday profile (5 min frequency data) for MSFT (other US & European liquid stocks gave similar results).

- Higher frequency results (1 min) were comparable. MLE procedure based on Dirichlet distribution was slow to converge (due to large dimension).
Gamma bridge captures main stylized facts (but misses idiosyncratic features as expected).

First moment very good fit, second moment less good - evidence against higher order polynomial approximation.
Stability of Parameters Across Time

- Illustrates 60 day rolling parameter calibrations for first 90 trading days of 2012.
- The key parameters are stable across time (important). Expected as this is capturing the intraday S-shape of the volume curve which is known to have a fixed shape.
The assumption of a time changed gamma bridge for relative volume makes a statement about the distributions of the observations in each time bucket.

It can be shown that the relative volume observation from each time bucket has a beta distribution with known parameters, namely $\gamma(t) \sim \text{Be}(\alpha(t), \beta(t))$ where

$$\alpha(t) = \hat{m}(\hat{a}t^3 + \hat{b}t^2 + (1 - \hat{a} - \hat{b})t) \quad \text{and} \quad \beta(t) = \hat{m}T - \alpha(t).$$

Use a 60 day rolling window to train parameters, then take 30 day out of sample test period.

In each bucket the sample mean is approximately Gaussian, with known mean & variance - Z-test. Additionally (for each bucket) the 30 observations come from a beta distribution - KS test.

Important trade-off, length of horizon vs non-stationarity of parameters.
Statistical Evaluation of the Fit

- Do not reject hypotheses in the morning but in the afternoon we do.
- This is not a multiple test (there is complicated distributional structure across the buckets).
- Conclusion - fit is acceptable for our purposes
Time Change in the Optimization

- To incorporate the intraday profile we time change the prices and relative volume process using $G$.
- Effectively we are assuming that volume and volatility profiles are approximately equivalent in normalised co-ordinates (reasonable).
- The time changed optimization can be connected to the original optimization problem - we keep the closed form solutions!
- Recall that our optimization problem is of the form

$$\min_u \mathbb{E}[S(u)] + \lambda \sqrt{\mathbb{V}[S(u)]}$$

- Given the optimal $\hat{u}$ we can look at the map (Markowitz)

$$\lambda \mapsto \left( \mathbb{E} [S(\hat{u})], \sqrt{\mathbb{V}[S(\hat{u})]} \right)$$
Statistical Evaluation of the Fit

- TWAP here is equivalent to historical mean.
- Expected slippage/deviation of slippage are of the correct magnitude.
- We can improve second moment, but not first moment of slippage performance.
Summary

- Formulated a tractable model of relative volume based on a gamma bridge which captures the stylized facts of the data.
- For this first time incorporated this into a continuous time framework allowing a mean-variance analysis of VWAP execution including market impact.
- Provides accurate and realistic estimates for slippage & variance of slippage, gives an estimate of the best possible performance possible when tracking market VWAP.
- Indication is that we can improve the standard deviation of our execution - possible application in GVWAP.
Future Work

- Analyse estimators for relative volume and evaluate the effect on the optimal strategy where $\gamma$ is replaced by an estimator - can the control problem be solved in this case?
- Assign a value to knowledge of the terminal volume $V(T)$.
- Large trades?
- Increase the stock universe - less liquid, Asian names?
- Issue with different choice of $\kappa$ for VWAP and arrival price benchmarks. Is this a function of $\alpha$?
- Other algorithms - PctVol?
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