Implied Volatility of Leveraged ETF Options

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Leveraged Exchange Traded Funds (LETFs) promise a fixed leverage ratio with respect to a given underlying asset or index.

Most typical leverage ratios are: (long) $1, 2, 3$, and (short) $-1, -2, -3$.

Significant rise in the use of (L)ETFs and their options in the past decade.

In 2012, total notional on ETF options is $40-50$ bil, as compared to $90$ bil for S&P 500 index options.

ETF options across 4 asset classes: equity, debt, commodity, and currency.

85% of the total ETF options volume is traded among only four ETFs: SPY (S&P 500), IWM (Russell 2000), QQQ (Nasdaq 100) and GLD (gold).
Empirical LEFT Prices

Figure: SSO and SDS cumulative returns from Dec 2010 to Nov 2011. Observe that both SSO and SDS can give negative returns simultaneously over several periods in time.
Figure: 1-day (left), 2-week (mid) and 2-month (right) returns of SPY against SSO (top) and SDS (bottom), in logarithmic scale. We considered 1-day, 2-week and 2-month rolling periods from Sept 29, 2010 to Sept 30, 2012.
Figure: 1-day (left), 2-week (mid) and 2-month (right) returns of SPY against UPRO (top) and SPXU (bottom), in logarithmic scale. We considered 1-day, 2-week and 2-month rolling periods from Sept 29, 2010 to Sept 30, 2012.
LETF Price Dynamics in the Black-Scholes Model

- Under the unique risk-neutral measure $\mathbb{P}^*$, the reference asset price follows:

$$\frac{dX_t}{X_t} = r\ dt + \sigma\ dW^*,$$

with constant interest rate $r$ and constant volatility $\sigma$.

- A long LETF $L$ on $X$ with leverage ratio $\beta \geq 1$ is constructed by
  - investing the amount $\beta L_t$ ($\beta$ times the fund value) in $X$,
  - borrowing the amount $(\beta - 1)L_t$ at the risk-free rate $r$,
  - expense charge at rate $c$.

- For a short ($\beta \leq -1$) LETF, $|\beta|L_t$ is shorted on $X$, and $(1 - \beta)L_t$ is kept in the money market account.

- The LETF price dynamics:

$$\frac{dL_t}{L_t} = \beta \left( \frac{dX_t}{X_t} \right) - ((\beta - 1)r + c)\ dt$$

$$= (r - c)\ dt + \beta \sigma\ dW^*_t.$$
The LETF value $L$ can be written in terms of $X$:

$$\frac{L_t}{L_0} = \left( \frac{X_t}{X_0} \right)^\beta e^{-(r(\beta-1)+c)t - \frac{\beta(\beta-1)}{2} \sigma^2 t}.$$

Over longer times, the volatility will lead to significant attrition in fund value, even if the underlying is performing well.

The no-arbitrage price a European call option on $L$ is:

$$C^{(\beta)}_{BS}(t, L; K, T) = e^{-r(T-t)} \mathbb{E}^*\{(L_T - K)^+ | L_t = L\} = C_{BS}(t, L; K, T, r, c, |\beta|\sigma),$$

where $C_{BS}(t, L; K, T, r, c, \sigma)$ is the Black-Scholes formula for a call.
Implied Volatility (IV) of LETF Options

- Given the market price $C^{\text{obs}}$ of a call on $L$, the implied volatility is given by

$$I^{(\beta)}(K, T) = (C_{BS}^{(\beta)})^{-1}(C^{\text{obs}}) = \frac{1}{|\beta|}C_{BS}^{-1}(C^{\text{obs}}).$$

- We normalize by the $|\beta|^{-1}$ factor in our definition of implied volatility so that they remain on the same scale.

**Proposition**

The slope of the implied volatility curve admits the bound:

$$-\frac{e^{-(r-c)(T-t)}}{|\beta|L\sqrt{T-t}} \frac{1 - N(d_2^{(\beta)})}{N'(d_1^{(\beta)})} \leq \frac{\partial I^{(\beta)}(K)}{\partial K} \leq \frac{e^{-(r-c)(T-t)}}{|\beta|L\sqrt{T-t}} \frac{N(d_2^{(\beta)})}{N'(d_1^{(\beta)})},$$

where $d_2^{(\beta)} = d_1^{(\beta)} - |\beta|I^{(\beta)}(K)\sqrt{T-t}$, with $\sigma = I^{(\beta)}(K)$. 
Empirical Implied Volatilities – SPX, SPY

Figure: SPX (blue cross) and SPY (red circles) implied volatilities on Sept 1, 2010 for different maturities (from 17 to 472 days) plotted against log-moneyness:

$$LM = \log \left( \frac{\text{strike}}{(L)\text{ETF price}} \right).$$
Empirical Implied Volatilities – SPY, SSO

Figure: SPY (blue cross) and SSO (red circles) implied volatilities against log-moneyness.
Empirical Implied Volatilities – SPY, SSO

Figure: SPY (blue cross) and SSO (red circles) implied volatilities against log-moneyness.
Empirical Implied Volatilities – SPY, SSO

Figure: [Left] SPY (blue cross) and SSO (red circles) implied volatilities against log-moneyness. [Right] SSO:SPY implied volatility ratios for different maturities.
Figure: SPY (blue cross) and SDS (red circles) implied volatilities against log-moneyness (LM) for increasing maturities.
Empirical Implied Volatilities – SPY, SDS

Figure: *SPY (blue cross) and SDS (red circles) implied volatilities against log-moneyness (LM) for increasing maturities.*
Empirical Implied Volatilities – SPY, SDS

Figure: [Left] SPY (blue cross) and SDS (red circles) implied volatilities against log-moneyness. [Right] SDS:SPY implied volatility ratios for different maturities.
Observations

- The most salient features of the empirical implied volatilities:
  - IV skew for SSO appears to be flatter than that for SPY,
  - IV skew is upward sloping for SDS, and downward sloping for SPY and SSO.
- Also, intuitively,
  - a put on a long-LETF and a call on a short-LETF are both bearish,
  - IVs should be higher for smaller (larger) LM for long (short) LETF.

- Traditionally, IV is used to compare option contracts across strikes & maturities. What about across leverage ratios?
- Which pair of LETF options should have comparable IV?
Observations

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  - IV skew for SSO appears to be flatter than that for SPY,
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- Which pair of LETF options should have comparable IV?
Multiscale Stochastic Volatility Framework

We assume that the reference index $X$, the LETF $L$, fast volatility factor $Y$ and slow volatility factor $Z$ are described by the system of SDEs:

\[
\begin{align*}
  dX_t &= rX_t \, dt + f(Y_t, Z_t)X_t \, dW_t^{(0)} \\
  dL_t &= (r - c)L_t \, dt + \beta f(Y_t, Z_t)L_t \, dW_t^{(0)} \\
  dY_t &= \left( \frac{1}{\varepsilon} \alpha(Y_t) - \frac{1}{\sqrt{\varepsilon}} \eta(Y_t)\Lambda_1(Y_t, Z_t) \right) \, dt + \frac{1}{\sqrt{\varepsilon}} \eta(Y_t) \, dW_t^{(1)} \\
  dZ_t &= \left( \delta \ell(Z_t) - \sqrt{\delta} \, g(Z_t)\Lambda_2(Y_t, Z_t) \right) \, dt + \sqrt{\delta} \, g(Z_t) \, dW_t^{(2)} .
\end{align*}
\]

Here, the standard $\mathcal{B}^\star$-Brownian motions $(W^{(0)}\!, W^{(1)}\!, W^{(2)}\!)$ are correlated:

\[
\begin{align*}
  d\langle W^{(0)}, W^{(1)} \rangle_t &= \rho_1 \, dt, \\
  d\langle W^{(0)}, W^{(2)} \rangle_t &= \rho_2 \, dt, \\
  d\langle W^{(1)}, W^{(2)} \rangle_t &= \rho_{12} \, dt ,
\end{align*}
\]

where $|\rho_1|, |\rho_2|, |\rho_{12}| < 1$, and $1 + 2\rho_1\rho_2\rho_{12} - \rho_1^2 - \rho_2^2 - \rho_{12}^2 > 0$.

We call $\Lambda_1(y, z)$ and $\Lambda_2(y, z)$ the market prices of volatility risk.
ETF Option Price Approximation

The no arbitrage price of a European option on $X$ is given by

$$P^{\varepsilon, \delta}(t, X_t, Y_t, Z_t) = I E^* \left\{ e^{-r(T-t)} h(X_T) \mid X_t, Y_t, Z_t \right\}.$$  

Instead of solving the high-dim PDE, we apply the perturbation theory, as discussed in Fouque et al. (2011), to simplify and study the calibration problem.

**Proposition**

*For fixed $(t, x, y, z)$, the European option price $P^{\varepsilon, \delta}(t, x, y, z)$ is approximated by $P^*(t, x, z)$, where*

$$P^* = P_{BS}^* + \left\{ \tau V_0^\delta + \tau V_1^\delta \left( x \frac{\partial}{\partial x} \right) + \frac{V_3^\varepsilon}{\sigma^*} \left( x \frac{\partial}{\partial x} \right) \right\} \frac{\partial P_{BS}^*}{\partial \sigma},$$

*and the order of accuracy is given by*

$$P^{\varepsilon, \delta} = P^* + O(\varepsilon \log |\varepsilon| + \delta).$$

*Here, $P_{BS}^*$ is the Black-Scholes call price with time-to-maturity $\tau = T - t$ and the corrected volatility parameter $\sigma^*$.*
ETF Implied Volatility Approximation

- Using the price approximation, we can derive the 1st-order approximation for the IV.
- To do so, we define the variable *Log-Moneyness to Maturity Ratio* by

\[
\text{LMMR} = \frac{\log(K/x)}{\tau}.
\]

**Proposition**

The 1st-order approximation for the IV is given by

\[
I = b^* + \tau b^\delta + (a^\varepsilon + \tau a^\delta) \text{LMMR} + \mathcal{O}(\varepsilon \log |\varepsilon| + \delta),
\]

where \((b^*, b^\delta, a^\varepsilon, a^\delta)\) are defined by

\[
\begin{align*}
b^* &= \sigma^* + \frac{V_3^\varepsilon}{2\sigma^*} \left(1 - \frac{2r}{\sigma^*^2}\right), \quad a^\varepsilon = \frac{V_3^\varepsilon}{\sigma^*^3}, \\
b^\delta &= V_0^\delta + \frac{V_1^\delta}{2} \left(1 - \frac{2r}{\sigma^*^2}\right), \quad a^\delta = \frac{V_1^\delta}{\sigma^*^2}.
\end{align*}
\]
LET F Options Price Approximation

- The no-arb. price of a European LETF option on $L$ is given by
  \[ P_{\beta}^{\varepsilon,\delta}(t, L_t, Y_t, Z_t) = \mathbb{E}^\star \left\{ e^{-r(T-t)} h(L_T) \mid L_t, Y_t, Z_t \right\}. \]

- **Goal**: examine the role of $\beta$ in the coeff. of LETF option price asymptotics.

**Proposition**

For fixed $(t, x, y, z)$, the LETF option price $P_{\beta}^{\varepsilon,\delta}(t, x, y, z)$ is approximated by

\[ P_{\beta}^{\varepsilon,\delta} = P_{\beta}^* + O(\varepsilon \log |\varepsilon| + \delta), \]

where

\[ P_{\beta}^* = P_{BS}^* + \left\{ \tau V_{0,\beta}^{\delta} + \tau V_{1,\beta}^{\delta} \left( x \frac{\partial}{\partial x} \right) + \frac{V_{3,\beta}^{\varepsilon}}{\sigma_{\beta}^*} \left( x \frac{\partial}{\partial x} \right) \right\} \frac{\partial P_{BS}^*}{\partial \sigma}. \]

Here the $\beta$-dependent group market parameters are:

\[ V_{0,\beta}^{\delta} = |\beta| V_{0}^{\delta}, \quad V_{1,\beta}^{\delta} = \beta |\beta| V_{1}^{\delta}, \quad V_{3,\beta}^{\varepsilon} = \beta^3 V_{3}^{\varepsilon}, \quad \sigma_{\beta}^* = |\beta| \sigma^*. \]
The IV of an LETF option $I^{(\beta)}$ is defined by

$$I^{(\beta)} = \frac{1}{|\beta|} P_{BS}(P_{\beta}).$$

How does $\beta$ impact the 1st-order approximation of the IV?

**Proposition**

The 1st-order approximation of $I^{(\beta)}$ is given by

$$I^{(\beta)} \approx b^*_\beta + \tau b^\delta_\beta + (a^\varepsilon_\beta + \tau a^\delta_\beta) \text{ LMMR},$$

where the skew slopes are given in terms of the unleveraged ETF skew slopes by

$$a^\varepsilon_\beta = \frac{1}{\beta} a^\varepsilon, \quad a^\delta_\beta = \frac{1}{\beta} a^\delta,$$

and where the level parameters $(b^*_\beta, b^\delta_\beta)$ are

$$b^*_\beta = \sigma^* + \frac{\beta V^\varepsilon_3}{2\sigma^*} \left( 1 - \frac{2r}{\beta^2 \sigma^*} \right), \quad b^\delta_\beta = V^\delta_0 + \frac{\beta V^\delta_1}{2} \left( 1 - \frac{2r}{\beta^2 \sigma^*} \right).$$
Predicting from SPY IVs to LETF IVs

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Predicting from SPY IVs to LETF IVs

- SSO
- SDS
- UPRO
- SPXU

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Discrepancy of Calibrated IV Slope & Intercept, 9/09–9/10

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Implied Volatility of Leveraged ETF Options
IV Approximation for All Maturities

Figure: LETF options IVs and their calibrated first-order approximations for all available maturities on August 23, 2010.
IV Approximation for All Maturities

Figure: LETF options IVs and their calibrated first-order approximations for all available maturities on August 23, 2010.
Examining the Slope Relationships

- We calibrate daily the skew parameters \((a_\beta^\varepsilon, a_\beta^\delta)\) for \(\beta \in \{1, \pm 2, \pm 3\}\), and compute the ratio

\[
R_{ij} := \frac{a_{\beta_i}^\varepsilon + \tau a_{\beta_i}^\delta}{a_{\beta_j}^\varepsilon + \tau a_{\beta_j}^\delta}, \quad i, j \in \{1, \pm 2, \pm 3\}, \ i \neq j,
\]

for the 10 pairwise combinations. In theory, we should have

\[
R_{ij} = \frac{\beta_j}{\beta_i}.
\]

- Anticipating this might not hold precisely in the data, we look for any systematic deviation from these relationships.
- For each leverage, we calculate estimates of each \(\beta_i\) as if \(\beta_j\) were correct, and then average over the four estimates for each \(\beta_i\).
- That is, for each daily observation \(R_{ij}\), we compute the estimates

\[
\hat\beta_{j|i} := \beta_i R_{ij}, \quad \hat\beta_{i|j} := \beta_j / R_{ij},
\]

and take average to get

\[
\bar\beta_i = \frac{1}{4} \sum_{j \neq i} \hat\beta_{i|j}.
\]
Figure: Implied $\beta$: Histograms of $\bar{\beta}_i$ from 9/09-9/10 LETF IVs.
Average Implied $\beta$’s from IVs

<table>
<thead>
<tr>
<th></th>
<th>SPY</th>
<th>SSO</th>
<th>SDS</th>
<th>UPRO</th>
<th>SPXU</th>
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<tbody>
<tr>
<td>$\beta$ mean</td>
<td>1.0512</td>
<td>1.9840</td>
<td>-2.6314</td>
<td>2.4705</td>
<td>-2.9264</td>
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<tr>
<td>$\beta$ standard dev.</td>
<td>0.1092</td>
<td>0.1596</td>
<td>0.5101</td>
<td>0.2605</td>
<td>0.4780</td>
</tr>
</tbody>
</table>

Table: The mean and standard deviation of estimated $\bar{\beta}$ over 9/09–9/10.

- SPY, SSO and, surprisingly, SPXU implied volatilities are consistent with their leverage ratios of 1, 2 and $-3$ respectively.
- IV skews from SDS ($\beta = -2$) systematically overestimate the magnitude of leverage ratio, as if the LETF was more short than it is actually supposed to be.
- Skews from UPRO ($\beta = 3$) systematically underestimate the leverage ratio, as if the LETF was not so ultra-leveraged.
Moneyness Scaling

- To link the IVs between ETF and LETF options, we introduce the method of moneyness scaling.
- In a general stochastic volatility model, we can write the log LETF price as

\[
\log \left( \frac{L_T}{L_0} \right) = \beta \log \left( \frac{X_T}{X_0} \right) - (r(\beta - 1) + c)T - \frac{\beta(\beta - 1)}{2} \int_0^T \sigma_t^2 dt.
\]

- Conditioning on that the terminal (random) log-moneyness \( \log \left( \frac{X_T}{X_0} \right) \) equal to constant \( LM^{(1)} \), we compute the cond’l expectation (best estimate) of the \( \beta \)-LETF log-moneyness.
- This leads us to define

\[
LM^{(\beta)} := \beta LM^{(1)} - (r(\beta - 1) + c)T - \frac{\beta(\beta - 1)}{2} \mathbb{E}^* \left\{ \int_0^T \sigma_t^2 dt \mid \log \left( \frac{X_T}{X_0} \right) = LM^{(1)} \right\}.
\]
Moneyness Scaling

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Conditioning on that the terminal (random) log-moneyness $\log \left( \frac{X_T}{X_0} \right)$ equal to constant $LM^{(1)}$, we compute the cond’l expectation (best estimate) of the $\beta$-LETF log-moneyness.

This leads us to define

$$LM^{(\beta)} := \beta LM^{(1)} - (r(\beta - 1) + c)T - \frac{\beta(\beta - 1)}{2} \mathbb{E}^* \left\{ \int_0^T \sigma_t^2 dt \mid \log \left( \frac{X_T}{X_0} \right) = LM^{(1)} \right\}.$$
Connecting Log-moneyness

- Assuming constant $\sigma$ as in the B-S model, we have the formula:

$$LM^{(\beta)} = \beta LM^{(1)} - (r(\beta - 1) + c)T - \frac{\beta(\beta - 1)}{2}\sigma^2 T.$$  

- Hence, the $\beta$-LETF log-moneyness $LM^{(\beta)}$ is expressed as an affine function of the unleveraged ETF log-moneyness $LM^{(1)}$, reflecting the role of $\beta$.

- The moneyness scaling formula can be interpreted via Delta matching.

**Proposition**

*Under the B-S model, an ETF call with $LM^{(1)}$ and a $\beta$-LETF call with $LM^{(\beta)}$ have the same Delta if and only if (1) holds.*
Simulated IV Comparison via Moneyness Scaling

Figure: [Left] As $\beta$ increases from 1 to 3, the IV skew becomes visibly flatter. [Right] After moneyness scaling, the IVs are significantly closer.
Empirical IV Comparison via Moneyness Scaling

Figure: *SPY* (blue cross) and *LETF* (red circles) implied volatilities after moneyness scaling on Sept 1, 2010 with 108 days to maturity, plotted against log-moneyness of *SPY* options.
Empirical IV Comparison via Moneyness Scaling

Figure: SPY (blue cross) and LETF (red circles) implied volatilities after moneyness scaling on Sept 1, 2010 with 108 days to maturity, plotted against log-moneyness of SPY options.
Concluding Remarks

- We have discussed a stochastic volatility framework to understand the inter-connectedness of LETF options.
- The method of moneyness scaling enhances the comparison of IVs with different leverage ratios.
- The connection allows us to use the richer unleveraged index/ETF option data to shed light on the less liquid LETF options market.

Related problems: long or short time-to-expiration, liquidity, tracking errors, and feedback effect created by ETFs and LETFs.
Appendix
Assume $X$ follows the local volatility model:

$$\frac{dX_t}{X_t} = r\, dt + \sigma(t, X_t)\, dW^*_t.$$ 

Then the $\beta$-LETF $L$ follows

$$\frac{dL_t}{L_t} = (r - c)\, dt + \beta \sigma(t, X_t)\, dW^*_t.$$ 

or equivalently,

$$\frac{L_t}{L_0} = \left(\frac{X_t}{X_0}\right)^\beta\, e^{-(r(\beta-1)+c)t - \frac{\beta(\beta-1)}{2} \int_0^t \sigma(u, X_u)^2\, du}.$$ 

The realized variance term $\int_0^t \sigma(u, X_u)^2\, du$ means that $X_t$ and $L_t$ do not have one-to-one correspondence.

$L$ no longer follows a local volatility model.