<u>Thm</u> The following are equivalent for a proper flow f:

1. f is a maximum proper flow.
2. There are no GAPs in A_f.
3. $\exists$ labeling μ s.t. $\gamma^\mu(i,j) \le 1 \quad \forall (i,j) \in A_f$.

<u>Pf</u> Showed $(1) \Rightarrow (2)$, $(2) \Rightarrow (3)$ in class.

$(3) \Rightarrow (1)$ Spse we have a proper flow f, given labeling μ.

Consider any other proper flow $\tilde{f}$. Want to show $|f| \ge |\tilde{f}|$.

Compare $\tilde{f}^\mu$ and f^μ (but drop superscript μ).

Consider $(i,j) \in A$.

If $f(i,j) < \tilde{f}(i,j) \Rightarrow f(i,j) < u(i,j) \Rightarrow (i,j) \in A_f$

$\Rightarrow \gamma^\mu(i,j) \le 1$.

If $f(i,j) > \tilde{f}(i,j)$

$$-\gamma(j,i) f(j,i) > -\gamma(j,i)\tilde{f}(j,i)$$

$$f(j,i) < \tilde{f}(j,i) \le u(j,i)$$

$$\Rightarrow (j,i) \in A_f \Rightarrow \gamma^{\mu}(j,i) \le 1$$

$$\Rightarrow \gamma^{\mu}(i,j) \ge 1$$

$\therefore$ for any $(i,j) \in A$ $\quad (\gamma^{\mu}(i,j) - 1)(f(i,j) - \tilde{f}(i,j)) \ge 0$

$$\sum_{(i,j)\in A} (\gamma^{\mu}(i,j) - 1)(f(i,j) - \tilde{f}(i,j)) \ge 0$$

$$\Rightarrow \sum_{(i,j)\in A} \gamma^{\mu}(i,j)(f(i,j) - \tilde{f}(i,j)) - \sum_{(i,j)\in A} (f(i,j) - \tilde{f}(i,j)) \ge 0$$

$$\Rightarrow \sum_{(i,j)\in A} \left(\tilde{f}(j,i) - f(j,i)\right) - \sum_{(i,j)\in A} \left(f(i,j) - \tilde{f}(i,j)\right) \geq 0$$

Recall $e_f(i) = -\sum_{k:(i,k)\in A} f(i,k)$

$$\Rightarrow \sum_{i\in V} e_f(i) - \sum_{i \in V} e_{\tilde{f}}(i) \geq 0$$

Proper flow $f, \tilde{f}$, $e_f(i) = e_{\tilde{f}}(i) = 0 \quad \forall i \neq t.$

$$e_f(t) \geq e_{\tilde{f}}(t) \Rightarrow |f| \geq |\tilde{f}|.$$