ORIE 6300 Mathematical Programming I August 29, 2014

Problem Set 1

Due Date: September 5, 2014

1. (a) Consider the LP:

$$\begin{array}{rcl} \text{Min } c^T x + \bar{c}^T \bar{x} & & \\ Ax + \bar{A}\bar{x} & = & b \\ x & \geq & 0 \\ \bar{x} & & \text{unconstrained} \end{array}$$

Suppose we want to convert this problem to one in standard form; that is, with all variables being nonnegative. In class, we saw that one could do this by replacing the unconstrained variables with the difference of nonnegative variables (e.g. $\bar{x} = t - s$ for $s, t \geq 0$), but this doubles the number of such variables. Devise another technique to obtain an equivalent standard form problem where the number of variables is only increased by one.

(b) Consider the LP:

$$\begin{array}{rcl} \text{Max } y^T b & & \\ A^T y & \leq & c \\ \bar{A}^T y & = & \bar{c} \\ y & & \text{unconstrained} \end{array}$$

We want to convert this into a problem in the form of the dual to a standard form problem; i.e. with all less-than-or-equal-to constraints. The usual way to do this is to replace each equality constraint by two inequality constraints, but this doubles the number of such constraints. Devise another technique that only increase the number of constraints by one.

(c) What is the relationship between the techniques in (a) and (b)?

2. What is the dual of the linear program with variables $x \geq 0$ and an additional single variable λ with constraints $Ax \leq \lambda b$, where the objective is to minimize λ?

3. Consider the following LP:

$$\begin{array}{rcll} \text{Max } \sum_{i=1}^{n} v_i x_i & & & \\ \sum_{i=1}^{n} s_i x_i & \leq & B & \\ x_i & \leq & 1 & i = 1, \ldots, n \\ x_i & \geq & 0 & i = 1, \ldots, n \end{array}$$

(a) Show the dual of the LP above. Use the variable y_0 for the constraint with right-hand side B, and the variables y_i for the $x_i \le 1$ constraints.

(b) Assume that the numbers v_i and s_i are positive and that

$$\frac{v_1}{s_1} \ge \frac{v_2}{s_2} \ge \cdots \ge \frac{v_n}{s_n}.$$

Let k be the largest index such that $s_1 + s_2 + \cdots + s_{k-1} \le B$. Show that the following primal and dual LP solutions must be optimal:

$$x_i = \begin{cases} 1 & i < k \\ \frac{B-(s_1+s_2+\cdots+s_{k-1})}{s_k} & i = k \\ 0 & i > k \end{cases}$$

$$y_i = \begin{cases} \frac{v_k}{s_k} & i = 0 \\ s_i\left(\frac{v_i}{s_i} - \frac{v_k}{s_k}\right) & 0 < i < k \\ 0 & i \ge k \end{cases}$$