ORIE 6300 Mathematical Programming I	November 25, 2014
Lecture 26	
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1 Conic Programming

In the linear programming primal standard form, we have

$$\begin{aligned} \text{Inf } & c^T x \\ \text{subject to } & Ax = b \\ & x \geq 0. \end{aligned}$$

Now, with conic programming we replace this condition by requiring that $x \in K$ for some cone K. The primal form conic program is written as

$$\begin{aligned} \text{Inf } & c^T x \\ \text{subject to } & Ax = b \\ & x \in K. \end{aligned}$$

Recall a cone K is a set such that if $x, y \in K$ then $\lambda x + \mu y \in K$ for $\lambda, \mu \geq 0$.

2 Examples

There are some commonly used cones in conic programming, including

1. The non-negative orthant, $K = \{x \in \mathbb{R}^n : x \geq 0\}$.
2. The second-order cone, $K_{soc} = \{x \in \mathbb{R}^n : x_n^2 \geq \sum_{i=1}^{n-1} x_i^2, x_n \geq 0\}$, which is also called the Lorentz cone or the ice cream cone.
3. The positive semidefinite cone, $K = \{X \in \mathbb{R}^{n \times n} : X^T = X, v^T X v \geq 0 \forall v \in \mathbb{R}^n\}$. Notice that here we consider a set of matrices instead of vectors. In this case the conic program becomes

$$\begin{aligned} \text{Inf } & \sum_{i,j} c_{ij} x_{ij} \\ \text{subject to } & \sum_{ij} a_{ijk} x_{ij} = b_k, k = 1, ..., m \\ & X = (x_{ij}) \in K. \end{aligned}$$

Additionally, in conic programs notice how we now use the infimum instead of the minimum. The following example illustrates why. Consider

$$\begin{aligned}\text{Inf } & x_2 + x_3 \\ \text{subject to } & x_1 = 1 \\ & x \in K_{soc}.\end{aligned}$$

Since K_{soc} is the second order cone, we get that we have a feasible solution if and only if

$$x_3^2 \geq x_1^2 + x_2^2, \text{ and } x_3 \geq 0 \Leftrightarrow x_1^2 \leq (x_3 - x_2)(x_2 + x_3) \Leftrightarrow 1 \leq (x_3 - x_2)(x_2 + x_3).$$

So, for any $\epsilon > 0$, if we set $x_3 = \frac{1}{2}(\epsilon + \frac{1}{2})$ and $x_2 = \frac{1}{2}(\epsilon - \frac{1}{2})$, then we get $x_2 + x_3 = \epsilon$ and $x_3 - x_2 = \frac{1}{\epsilon}$. Therefore, the objective function $x_2 + x_3 = \epsilon$ can be arbitrarily small but cannot be zero.

3 Duality

For LP duality, we find a solution (y, s) s.t. $A^T y + s = c$ with $s \geq 0$. From this we obtain that

$$c^T x = (A^T y + s)^T x = y^T A x + s^T x = y^T b + s^T x \geq y^T b.$$

The last equality is because the primal requires $Ax = b$, and the last inequality is because the primal and dual solutions require $x, s \geq 0$. This gives us a lower bound on the optimal value of primal.

We apply the same sort of logic for the dual of the conic program. Here, we require that $x \in K^* \equiv \{s \in \mathbb{R}^n : s^T x \geq 0, \forall x \in K\}$. This gives us that the last inequality will still hold so that $y^T b$ is still a lower bound on the value of the primal. We get that the cone programming dual is

$$\begin{aligned}\text{Sup } & b^T y \\ \text{subject to } & A^T y + s = c \\ & s \in K^*.\end{aligned}$$

Thus by construction, we have that weak duality holds if this is the dual program.

K^* is the dual of the cone K, and if $K^* = K$ then the cone is called a **self-dual**. The non-negative orthant, second-order, and positive semidefinite cones are all self-dual, though we'll only prove this fact for the second-order cones.

Theorem 1 *K_{soc} is self-dual.*

Proof: First we'll show $K_{soc} \subseteq K_{soc}^*$. If $x, s \in K_{soc}$, then

$$x^T s = x_n s_n + \sum_{i=1}^{n-1} \geq x_n s_n - \sqrt{\sum_{i=1}^{n-1} x_i^2}\sqrt{\sum_{i=1}^{n-1} s_i^2} \geq 0.$$

The first inequality follows from the Cauchy-Schwartz inequality, and the last inequality follows from the fact that $x, s \in K_{soc}$.

Next we'll show $K_{soc}^* \subset K_{soc}$. Suppose $s^T x \geq 0 \forall x \in K_{soc}$. There are two cases. First, if $(s_1, ..., s_n - 1) = (0, ..., 0)$, then consider $x_1, ..., x_{n-1} = (0, ..., 0)$ and $x_n = 1$. Then we get that

$$s^T x \geq 0 \Leftrightarrow s_n \geq 0 \Leftrightarrow s_n^2 \geq \sum_{i=1}^{n-1} s_i^2 \Leftrightarrow s \in K_{soc}$$

Otherwise, set $x_n = \sqrt{\sum_{i=1}^{n-1} s_i^2}$ and $x_i = -s_i$ for $i = 1, .., n-1$, and so

$$s^T x \geq 0 \Leftrightarrow s_n \sqrt{\sum_{i=1}^{n-1} - \sum_{i=1}^{n-1} s_i^2 \geq 0} \Leftrightarrow s_n^2 \geq \sum_{i=1}^{n-1} s_i^2, s_n \geq 0 \Leftrightarrow s \in K_{soc}.$$

□

4 Weak and Strong Duality

We know that weak duality holds, but in general strong duality does not. Here are two examples showing that strong duality does not always hold.

Example 1: The primal has a finite value but the dual is not feasible. Consider the following primal/dual pair:

$$\begin{aligned} \text{Inf } & -x_1 \\ \text{subject to } & x_2 + x_3 = 0 \\ & x \in K_{soc} \end{aligned} \qquad\qquad \begin{aligned} & \text{Sup } 0y \\ & \text{subject to } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} y + s = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Notice that the constraint in the dual implies that $(-1, -y, -y)^T \in K_{soc}$.

In the primal, we have that

$$x_3^2 \geq x_1^2 + x_2^2 \quad \Leftrightarrow \quad (x_3 + x_2)(x_3 - x_2) \geq x_1^2 \quad \Leftrightarrow \quad 0 \geq x_1^2 \quad \Leftrightarrow \quad x_1 = 0,$$

and so an optimal primal solution is $(0, 0, 0)$ with a value of 0.

For the dual, we require

$$(-y)^2 \geq 1^2 + (-y)^2 \quad \Leftrightarrow \quad y^2 \geq 1 + y^2 \quad \Leftrightarrow \quad 0 \geq 1,$$

thereby showing that the dual is not feasible.

Example 2: Both the primal and the dual have finite, but different, values. Consider the following primal/dual pair:

$$\begin{aligned} \text{Inf } & -x_1 \\ \text{subject to } & x_1 + x_4 = 1 \\ & x_2 + x_3 = 0 \\ & x \in K \end{aligned} \qquad\qquad \begin{aligned} & \text{Sup } y_1 \\ & \text{subject to } -\begin{bmatrix} 1 + y_1 \\ y_2 \\ y_2 \\ y_1 \end{bmatrix} y \in K^* = K \end{aligned}$$

where $K = \{x \in \mathbb{R}^4 : x_3^2 \geq x_1^2 + x_2^2, x_3, x_4 \geq 0\}$, which is self-dual.

The primal is similar to the first example, so as before x_1 is forced to be 0. Therefore, an optimal solution is $(0, 0, 0, 1)$ of value 0. For the dual, we need

$$y_2^2 \geq y_2^2 + (-1 - y_1)^2 \text{ and } y_1, y_2 \leq 0 \Leftrightarrow (y_1 + 1)^2 \leq 0 \Leftrightarrow y_1 = -1 \text{ with } y_2 = 0.$$

Therefore, $(-1, 0)$ is an optimal solution with a value of -1.

Recalling that we proved strong duality holds using Farkas' Lemma, strong duality breaks down because Farkas' Lemma no longer holds. Recall that Farkas' Lemma states that exactly one the following is feasible:

1. $Ax = b, x \geq 0$
2. $A^T y \leq 0, b^T y > 0$

In the conic programming setting, we would get the analogous Farkas' Lemma stating that exactly one of the following holds:

1. $Ax = b, x \in K$
2. $-A^T y \in K^*, b^T y > 0$

However, we can show an example where neither of these hold. Consider

$$x_1 = 1, x_2 + x + 3 = 0, x \in K_{soc}.$$

In the first case we get

$$x_3^2 \geq x_1^2 + x_2^2 \quad \Leftrightarrow \quad (x_3 + x_2)(x_3 - x_2) \geq x_1^2 \quad \Leftrightarrow \quad 0 \geq 1,$$

and in the second case we get

$$-(y_1, y_2, y_2) \in K_{soc}, y_1 > 0, (-y_1)^2 + (-y_2)^2 \leq (-y_2)^2 \quad \Leftrightarrow \quad y_1^2 \leq 0 \quad \Leftrightarrow y_1 \leq 0.$$

Farkas' Lemma in conic programming is not true because the proof relied on applying the separating hyperplane theorem to $Q = \{b \in \mathbb{R}^m : \exists x \in \mathbb{R}^n, Ax = \hat{b}, x \geq 0\}$. We used that if $Ax = b, x \geq 0$ is not feasible, then $b \notin Q$, and finally applied the hyperplane theorem to obtain a y such that $A^T y \leq 0, b^T y > 0$.

But, recall that this can only be applied if Q is closed, nonempty, and convex. However, as a proof by counter example we show that the analogous Q in the conic programming setting does not need to be closed — consider

$$Q' = \left\{\hat{b} \in \mathbb{R}^2 : \exists x \in \mathbb{R}^3, s.t. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} \hat{b_1} \\ \hat{b_2} \end{bmatrix}, x \in K_{soc}\right\}.$$

Then Q' is nonempty since $(0, 0) \in Q'$. But, Q' is not closed since $(1, \epsilon) \in Q'$ for all $\epsilon > 0$, but $(1, 0) \notin Q'$.

This leads to the following idea: we say that $Ax = b, x \in K$ is **asymptotically feasible** if $\forall \epsilon > 0$, $\exists \Delta b$ such that $||\Delta b|| < \epsilon$ and $Ax = b + \Delta b, x \in K$ is feasible; that is, the system is always feasible if we can perturb the constraints a little, no matter how little we can perturb them. Next time we will prove the following theorem.

Theorem 2 *(Asymptotic Farkas Lemma) Either $Ax = b, x \in K$ is asymptotically feasible, or $-A^T y \in K^*, b^T y > 0$ is feasible, but not both.*