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Lecture 15

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1 Varieties of Simplex Method: Dual Simplex

1.1 Description

Recall that the regular (primal) simplex method is an algorithm that maintains primal feasibility and works towards dual feasibility. We start with a primal feasible solution and try to reach dual feasibility while maintaining complementary slackness. Dual simplex is exactly analogous to primal simplex where we start with a dual feasible solution corresponding to a basis B and move towards making the corresponding primal solution feasible while maintaining complementary slackness.

Consider the standard primal and dual linear programs.

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y \leq c \end{array}$$

Assume we have a dual basic feasible solution $y = (A_B^T)^{-1} c_B$ with associated basis B, then

$$A^T y \leq c \quad \text{i.e.} \quad \bar{c} \geq 0$$

Let

$$x_B = A_B^{-1} b = \bar{b}, \quad x_N = 0$$

If $\bar{b} \geq 0$, then x is primal feasible. Since x and y satisfy complementary slackness, they are primal and dual optimal solutions. If not, then there exists $i \in B$ such that $\bar{b}_i < 0$. So we want to remove i from the basis B. The next question is which index should we add to the basis.

Recall that the primal LP can be rewritten as

$$(1) \quad \begin{array}{ll} \min & c_B^T x_B + c_N^T x_N \\ \text{s.t.} & A_B x_B + A_N x_N = b \\ & x_B, x_N \geq 0 \end{array} \qquad \text{or} \qquad (2) \quad \begin{array}{ll} \min & c_B^T x_B + c_N^T x_N \\ \text{s.t.} & I x_B + \bar{A} x_N = \bar{b} \\ & x_B, x_N \geq 0 \end{array}$$

where $\bar{A} = A_B^{-1} A_N$, $\bar{b} = A_B^{-1} b$. Consider the i^{th} constraint of LP(2),

$$x_i + \sum_{j \in N} \bar{A}_{ij} x_j = \bar{b}_i < 0$$

If $\bar{A}_{ij} \geq 0, \forall j \in N$, since we also have $x \geq 0$, thus above constraint cannot be satisfied. Therefore there is no feasible solution to the primal LP in this case.

Now what should we do if there exists $j \in N$ such that $\bar{A}_{ij} < 0$? Taking the duals for both (1) and (2), we get

$$(3) \quad \begin{array}{ll} \max & b^T y \\ \text{s.t.} & A_B^T y \le c_B \\ & A_N^T y \le c_N \end{array} \qquad \text{or} \qquad (4) \quad \begin{array}{ll} \max & \bar{b}^T \tilde{y} \\ \text{s.t.} & I^T \tilde{y} \le c_B \\ & \bar{A}^T \tilde{y} \le c_N \end{array}$$

Note (3) and (4) are equivalent if we let $\tilde{y} = A_B^T y$. And we have already set y such that $A_B^T y = c_B$, or equivalently $\tilde{y} = c_B$. Consider LP (4). since $\bar{b}_i < 0$, we can increase the value of the objective function if we decrease $\tilde{y}_i$. But how far can we do this?

Suppose we decrease $\tilde{y}_i$ by δ. For any $j \in N$ such that $\bar{A}_{ij} \ge 0$, we still have $\bar{A}_j^T \tilde{y} \le c_j$. For any $j \in N$ such that $\bar{A}_{ij} < 0$, the LHS of the j^{th} constraint goes up by $-\bar{A}_{ij}\delta$. To stay feasible, we should have

$$\delta \le \frac{c_j - \bar{A}_j^T \tilde{y}}{-\bar{A}_{ij}} \quad \forall j \in N \text{ s.t. } -\bar{A}_{ij} < 0$$

We can rewrite it into a more familiar form

$$\delta \le \frac{c_j - \bar{A}_j^T \tilde{y}}{-\bar{A}_{ij}} = \frac{c_j - (A_B^{-1} A_j)^T \tilde{y}}{-\bar{A}_{ij}} = \frac{c_j - A_j^T (A_B^T)^{-1} \tilde{y}}{-\bar{A}_{ij}} = \frac{c_j - A_j^T y}{-\bar{A}_{ij}} = \frac{\bar{c}_j}{-\bar{A}_{ij}}$$

Therefore if we decrease $\tilde{y}_i$ by δ such that

$$\delta = \min_{j \in N : \bar{A}_{ij} < 0} \frac{\bar{c}_j}{-\bar{A}_{ij}}$$

then the dual variables are still feasible. And the index j that achieves this minimum will enter the basis.

1.2 Summary

In the dual simplex method,

- Set $x = A_B^{-1} b$, $y = (A_B^T)^{-1} c_B$. If $\bar{b} \ge 0$, then x and y are primal and dual optimal.
- (**Check for infeasibility**) Otherwise, $\bar{b}_i < 0$ for some i. If $\bar{A}_{ij} \ge 0$ for all $j \in N$, then the LP is primal infeasible.
- (**Ratio test**) Otherwise, $\bar{A}_{ij} < 0$ for some j. Compute

  $$\delta = \min_{j \in N : \bar{A}_{ij} < 0} \frac{\bar{c}_j}{-\bar{A}_{ij}}$$

  Pick $j \in N$ that attains the minimum as the index to add to the basis.
- (**Update basis**) $\hat{B} \leftarrow B - \{i\} \cup \{j\}$

1.3 Why use dual simplex?

- The dual simplex works better in practice.
  - It is usually easier to find initial dual feasible solutions.
    Since in practise we usually have $c \geq 0$, then $y = 0$ is a dual feasible solution.
  - The dual LP is often less degenerate.
- "Warm start"s: to solve another related LP after solving the first one.
  - If the objective function c changes, we use primal simplex with the previous primal optimum as an initial primal feasible solution.
  - If the RHS of the constraints b changes, we use dual simplex with the previous dual optimum as an initial dual feasible solution.
  - If an additional constraint is added to the primal LP, we use dual simplex.
    Now the dual LP has one more variable, if we set the new variable to be 0 and all other variables to be the previous dual optimum, we get a dual feasible solution for the new LP and can carry out dual simplex.
    This is frequently used in solving integer programming.

2 Sensitivity Analysis

In sensitivity analysis, we ask the question: How do solutions x and y change as input data (A, b, c) changes? We'll look at small, local changes of each of these in turn.

2.1 Changes in b

Suppose we increase b_i by δ. $b \to b + \delta e_i$ (e_i is a vector of 0s with 1 in i^{th} place.) Then $y = (A_B^T)^{-1} c_B$ stays the same and feasible. Let $x_N = 0$, $x_B = A_B^{-1}(b + \delta e_i)$, then complimentary slackness is still obeyed. If we also have $x_B \geq 0$, then x is feasible. Thus x and y are optimal solutions.

How does optimal objective funtion value change?

$$\triangle c = c_B^T(\triangle x_B) = \delta(c_B^T A_B^{-1} e_i) = \delta((A_B^T)^{-1} c_B)^T e_i = \delta y^T e_i = \delta y_i$$

So the optimal dual variable y_i gives the change in cost as we perturb the RHS b_i.
y_i is ***shadow price/marginal cost*** of b_i.

Now suppose we change b to $b \to b + \delta \hat{b}$. Then $y = (A_B^T)^{-1} c_B$ stays feasible. Let $x_N = 0$, $x_B = A_B^{-1}(b + \delta \hat{b})$. Then x and y stay optimal if $x_B \geq 0$. The objective function changes by

$$\triangle c = c_B^T(\triangle x_B) = \delta(c_B^T A_B^{-1} \hat{b}) = \delta((A_B^T)^{-1} c_B)^T \hat{b} = \delta y^T \hat{b}.$$

2.2 Changes in c

Suppose we change $c_j \to c_j + \delta$. Then x stays feasible. There are now two cases:

1. $j \in N$. The objective function stays the same, since $x_j = 0$. c_B is unchanged, so $y = (A_B^T)^{-1} c_B$ stays the same, and therefore complementary slackness still holds. Is y feasible? Yes, if $A_j^T y \leq c_j + \delta$. If so, then x, y are still optimal.

2. $j \in B$. Then $c_B \to c_B + \delta e_j$ and then $y = (A_B^T)^{-1} c_B \to \tilde{y} = (A_B^T)^{-1} c_B + \delta e_j)$. This new y is feasible if $A_k^T \tilde{y} \leq c_k$ for all $k \in \mathbb{N}$, i.e. if $A_k^T y + \delta {A^T}_k (A_B^T)^{-1} e_j \leq c_k$. So there are bounds on δ such that y stays feasible. Note that the objective function changes by δx_j.

2.3 Changes in A

Now suppose we change a single entry of the constraint matrix A; suppose $a_{ij} \to a_{ij} + \delta$. Once again there are two cases:

1. $j \in N$. In this case, x stays feasible (since $x_j = 0$). Then $y = (A_B^T)^{-1} c_B$ stays the same, so that complementary slackness still holds. y is feasible if $A_j^T y + \delta y_i \leq c_j$.

2. $j \in B$. In this case, both $x_B = {A_B}^{-1} b$ and $y = (A_B^T)^{-1} c_B$ change. We need to check whether $A_B + \delta e_i e_j^T$ remains singular! If so, we can solve for both x and y and check whether they remain feasible; if so, they will be optimal because they both still obey complementary slackness.